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WASHINGTON UNIVERSITY  
DEPT OF MATHEMATICS/ARTS AND SCIENCE  
ONE BROOKINGS DRIVE/CAMPUS BOX 1146  
ST. LOUIS, MI 631308. PERFORMING ORGANIZATION  
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13. ABSTRACT (Maximum 200 words)

A fundamental idea in Fourier analysis is that the Fourier Transform gives a simultaneous diagonalization of a small but very important class of operators including differentiation and integration. On the other hand, the Fourier Transform is not well suited for studying multiplication operators. The wavelet transform (and related transforms) give excellent simultaneous almost diagonalization of a very large class of operators which includes differentiation, integration, and multiplication: in fact, more generally singular integral operators and pseudo-differential operators. Professor Rochberg's recent work has been to use this fact to study such operators. Some work has been in the "real variable" tradition, other parts have involved operators on spaces of analytic functions.

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(Administered by AFSOR for DARPA)****Description of Work Since Previous  
Progress Report (April 12, 1992),  
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Professor Rochberg has continued his use of wavelets and related techniques in studying the behavior of operators acting on function spaces.

A fundamental idea in Fourier analysis is that the Fourier Transform gives a simultaneous diagonalization of a small but very important class of operators including differentiation and integration. On the other hand, the Fourier Transform is not well suited for studying multiplication operators. The wavelet transform (and related transforms) don't diagonalize any interesting classical operators. However these transforms give excellent simultaneous *almost diagonalization* of a very large class of operators which includes differentiation, integration, and multiplication; and in fact, more generally singular integral operators and pseudo-differential operators. Professor Rochberg's recent work has been to use this fact to study such operators. Some of the work has been in the "real variable" tradition, other parts have involved operators on spaces of analytic functions.

**Real Variable Results:** The idea of using NWO sequences (which are a variation on the idea of wavelets) to study operators on function spaces is outlined in the survey [R]. Because these techniques give simultaneous almost diagonalization of potential operators (i.e fractional order integration) and multiplication, the techniques can be used to study weighted norm estimates for potential operators. In addition to being of intrinsic interest, such results are known to be related to estimates for the eigenvalues of the Schödinger equation. These ideas are outlined in [R1] and developed in detail in [R2]. The same ideas can be taken further. The decomposition of functions induces a decomposition of operators. This can be used to write complicated operators as a sum of simple operators.

When such a decomposition of the operator is used with a Neumann series representation for eigenvectors a complicated representation is obtained for the eigenvector. This expression can then be analyzed using the hyperbolic geometry of the wavelet phase space. The result is pointwise estimates for eigenvectors of singular integral operators and for solutions of certain differential equations with rough coefficients. This is presented in [R3].

**Complex Variable Results:** Wavelets, as well as related ideas such as using the Bergman kernel function or using molecular decompositions in spaces of holomorphic functions, give good approximate diagonalization of the Bergman and the Cauchy-Szego projection as well as of multiplication operators. Hence those ideas can be used effectively to study Hankel and Toeplitz operators which are built by mixing multiplication and projection. Those tools can also be used to obtain sparse (although not approximately diagonal) realizations of composition operators on spaces of holomorphic functions. Much of Rochberg's techniques to study Hankel operators, Toeplitz operators, and composition operators on spaces of holomorphic functions.

Some of this work done with my former student Z. Wu. This work, mentioned briefly in the previous progress report is described in [RW1] and [RW2].

With my current student, Peng Lin, I am investigating Hankel operators on spaces with weights. This has resulted in the manuscripts [LR1] and [RW2].

My student Hong Xian is studying composition operators. The starting point for his investigation was an attempt to find an analog of the pointwise eigenvector estimates of [R3] in the context of composition on the Hardy space. The project evolved and we have found a number of interesting relations between the size of composition operators and the size of their principal eigenvector. A manuscript is currently being prepared.

Wavelets and related techniques would also be expected to give sparse representations of operators made by mixing multiplication and composition operators. I am currently supervising the research of J. Wang on the properties of this class of operators. This class of operators is a natural extension of the class which I studied in [R4].

**Cancellation:** A recurrent theme in both the real and complex variable

work is the appearance of certain subtle cancellations. In the real variable work the cancellation shows up in commutator and paraproduct estimates. In the complex variable result it shows up in the boundedness criteria for Hankel operators. Also, it has recently been noted that this cancellation is closely related to the theory of compensated compactness. Recently I have been exploring this type of cancellation in the context of singular integral operators [GR], Hankel operators [PR], and interpolation theory [R5]. Although I have been able to obtain results in each context, the underlying general pattern remains elusive and is one of the themes of my current work.

Professors Weiss and Taibleson have continued their work on the  $\phi$ -transform and  $\psi$ -transform and the characterization of wavelets in terms of the Fourier transform. Together with one of their students, Xihua Wang, and post doctoral fellow Xiang Fang they have obtained several characterizations of classes of wavelets.

It is well known that the Fourier transform of an orthonormal wavelet must be supported on a set of measure at least  $2\pi$ . When the measure of the support equals  $2\pi$ , the wavelet in question is called MSF wavelet (minimal supported frequency wavelet). They have shown that all these wavelets are characterized by a rather simple equation. Moreover, they have a description of those wavelets that do and those that do not arise from multi-resolution analysis. Moreover, certain algorithms have been developed that enable us to obtain MSF Wavelets that are not band limited. This work is now being prepared for publication.

In collaboration with E. Berkson of the University of Illinois and M. Paluszynski, a post doctoral fellow of the University of Wroclaw, Weiss has obtained a wavelet characterization of all multiplier operators on  $L^p(\mathbb{R}^n)$ .

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**Progress Report for AFSOR Grant 90-0323  
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**Description of Work Done Through April 13, 1992**

Professors Rochberg, Taibleson, and Weiss have been working in various different areas related to Wavelet Theory.

Professor Rochberg's work is mostly concentrated in the study of classical spaces of functions and operators on those spaces. The unifying theme which connects the topics is the central role played, explicitly or implicitly, by group actions, generally the translation and dilation group ("ax + b group") which is at the heart of wavelet analysis. More specifically his work lies in three areas; Nearly Weakly Orthonormal (NWO) sequences, Calderón-Toeplitz operators, and Dirichlet Spaces of Analytic Functions.

**NWO Sequences:** These are sequences of functions in  $L^2(\mathbb{R}^n)$  in which each function is, roughly, localized at a dyadic cube. The functions are allowed to be a bit larger and rougher than wavelets but they play a very similar role in certain decompositions of functions in classical function spaces [RS], [R1]. In particular, large classes of singular integral operators can be written as sums of rank one operators associated with NWO sequences. This gives a very useful tool for obtaining boundedness criteria and eigenvalue estimates. He has recently been using NWO sequences as a tool in obtaining boundedness and eigenvalue estimates for large classes of weighted potential operators and for the Schrödinger operator [R2]. This approach gives most of the known results easily and quickly as well as new results. These ideas are taken further in [R3] where he studies the size of eigenvectors for certain singular integral operators.

Those results, which are for a relatively simple model operator, are very suggestive and he hopes to be able to obtain similar results for the eigenvectors of a large class of operators including the Schrödinger operator. He also plans to use NWO sequences to study the generalized eigenvectors of the Schrödinger operator (i.e. functions which are not in  $L^2$  but do satisfy the eigenvalue equation) by working with the Born series (Neumann series) associated with the Lipman-Schwinger equation. The kernel for the relevant integral operator in  $\mathbb{R}^3$  is

$$v^{1/2}(x)v^{1/2}(y)\exp(ik|x-y|)|x-y|^{-1}$$

(here  $v(x)$  is the potential function for the Schrödinger equation). To use NWO sequences here, the technique must be modified to deal with oscillatory kernels.

**Calderón-Toeplitz operators:** Inserting multiplication operator into the Bergman reproducing formula on the Bergman space produces Toeplitz operators on the Bergman space. Inserting a multiplication operator in the Calderón reproducing formula (continuous wavelet transformation) on  $L^2(\mathbb{R}^n)$  produces a class of operators which, by analogy, we call Calderón-Toeplitz operators. These operators are Calderón-Zygmund operators but can be studied by techniques quite similar to those used to study Toeplitz operators on the Bergman space. In [R4,5] Rochberg establishes a correspondence principle for these operators which says that there is a correspondence between the number of eigenvalues of size  $\lambda$  and the volume of the set in phase space on which the symbol is close to  $\lambda$ . The study of this class of operators is taken much further in the

thesis of one of our doctoral student, K. Nowak, who recently completed his studies. Some of his work is contained in [N1,2,3]. In addition to giving new results for commutators with singular integral operators, his results also give a new "weak 1-1" endpoint result for Hankel operators on the Bergman space. That result is the crucial step in a new and simple proof of the earlier  $S_p$  results for those operators.

**Dirichlet Spaces of Analytic Functions:** Although wavelets can be used to obtain decomposition theorems for classical function spaces and to study operators on those spaces, it is often more convenient to use reproducing kernels instead of wavelets. The theories for the two types of decompositions are generally quite similar. Decomposition theorems based on reproducing kernels are known to be quite useful on the Hardy and Bergman spaces [R6]. Recently, in collaboration with Z. Wu, Rochberg has been working on similar analysis on the Dirichlet space [RW1,2]. Fundamental new difficulties arise in this context due to the potential theory associated with the Dirichlet space. For instance, the analog of Carleson measures for the Dirichlet space cannot be given a purely local characterization. However, one is able to obtain decomposition theorems for the space of Carleson measures associated to the Dirichlet space and these results can be used to study operators on the Dirichlet space.

Professor Taibleson is engaged in the construction of libraries of bases for p-series bases. This work generalizes the 2-series bases built around the Haar/Walsh libraries that are in current use. It is hoped that this may result in finer resolution (by use of a grey scale).

Professor's Taibleson and Rochberg will be continuing their work with their student, Susan Kelly, to develop analytic work to analyze Gibbs's effects for wavelet expansions. Of particular interest is the behaviour of the Gibbs's effect for jump discontinuities at irrational points. The Gibbs's effect at non-dyadic rational points is fairly well understood but it may be chaotic at irrational points.

Taibleson, Rochberg, and Weiss will be working jointly with Wang Xihua and Susan Kelly on a development of graphical realizations of the Gibbs's effects as well as the development of efficient programs for computing the magnitude of the Gibbs's effects.

Professor Weiss, in collaboration with A. Bonami and F. Soria [BSW] has obtained necessary and sufficient conditions for a function  $\psi$  to be a band-limited orthogonal wavelet. More specifically, under some natural hypothesis on the Fourier transform,  $\hat{\psi}$ , of  $\psi$ , these conditions determine precisely when

$$\{\psi_{j,k}(x)\} = \{2^{j/2}\psi(2^j x - k)\}, j, k \in \mathbb{Z},$$

is an orthonormal basis for  $L^2(\mathbb{R})$ . One interesting aspect of these conditions is that  $b = |\hat{\psi}|$  is precisely one of the bell functions associated with a local sine/cosine basis of Coifman and Meyer (see [AWW]). The system  $\{\psi_{j,k}\}$  is then made of dilates of such local bases. Professor Weiss has also continued his research on the extension of the  $\psi$ -transform methods to spaces of homogeneous type (see [HJTW], [HW]).

Research into algorithms developed at Yale University has been continued at Washington University by M.V. Wickerhauser and his collaborators: speech signal segmentation, one- and two-dimensional best-basis transform image compression algorithms, and computation with wavelets and wavelet packets.

One difficulty with local sine and cosine bases as described in [CW], [AWW] is that the basis elements are not uniformly well-localized in time-frequency. In the continuum limit the product of time uncertainty and frequency uncertainty becomes infinite. In the finite approximation, this means lower compression ratios and poorer frequency localization for long windows.

Postdoctoral researcher Xiang Fang wrote an X-windows based application to take acoustic signals and segment them by the "multiple folding" version of the adapted local sine transform [CFS]. This adaptation uses windows with side steepness adjusted to their width so as to preserve good time-frequency localization. The convenient interface for the library of subroutines written by Fang in 1991 will speed up experiments in

speech signal recognition. Fang has also performed several experiments in image compression and restoration using the same library and is writing an interface for that as well.

M.V. Wickerhauser has prepared a one-dimensional command-line version of the fast approximate Karhunen-Loeve algorithm (the 2-dimensional version is described in [W11]. It replaces an oscillatory input signal with a small number of parameters, the principle orthogonal factors in an ensemble of test signals. This serves as a front-end for a one-dimensional signal classification device, which is currently undergoing tests at a private company. One other application of this algorithm, a cooperative effort with the Atmospheric Radiation Analysis Laboratory (Ecole Polytechnique/European Space Agency), is awaiting a large transfer of data from France.

In a series of papers on signal and image processing ([W12], [W13], [CMW]), we have described a new approach to signal processing, showing that signal analysis and data compression are related problems and solvable by similar methods. The algorithms behind these transforms are being professionally implemented for several computing platforms.

An expository article [AWW] describes the relationship between the local cosine and local sine bases and the smooth orthogonal wavelet transform. Although the main ideas come from another group, we feel that a detailed exposition by implementers who are familiar with the technical details has a separate value even beside the original articles.

Collaboration with M. Farge of the Ecole Normale Supérieure resulted in the discovery of a qualitatively better method for rank reduction in pseudospectral Galerkin decaying 2-D turbulence simulations [FGMPW]. This is the first stage of an experimental mathematics project to achieve rank reduction (and therefore lower complexity) in such numerical simulations. Further development of numerical algorithms has begun with methods to apply and multiply operators in the wavelet packet or best-basis [W14].